Exercise 20

A spring with a mass of 2 kg has damping constant 16, and a force of 12.8 N keeps the spring stretched 0.2 m beyond its natural length. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2.4 m/s.

Solution

Use the fact that a 12.8 N force stretches the spring 0.2 m to determine the spring constant k.

$$F = k(x - x_0)$$

$$12.8 \text{ N} = k(0.2 \text{ m})$$

$$k = 64 \, \frac{\text{N}}{\text{m}}$$

The initial value problem for position is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0, \quad x(0) = 0, \quad x'(0) = 2.4.$$

Because it has constant coefficients, it has solutions of the form $x = e^{rt}$.

$$x = e^{rt} \quad \rightarrow \quad \frac{dx}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x}{dt^2} = r^2e^{rt}$$

Substitute these formulas into the ODE.

$$m(r^2e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + cr + k = 0$$

Solve for r, noting that with the given values $c^2 - 4mk < 0$.

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c \pm i\sqrt{4mk - c^2}}{2m}$$

$$r = \left\{ \frac{-c - i\sqrt{4mk - c^2}}{2m}, \frac{-c + i\sqrt{4mk - c^2}}{2m} \right\}$$

Two solutions to equation (1) are

$$\exp\left(\frac{-c - i\sqrt{4mk - c^2}}{2m}t\right) \quad \text{and} \quad \exp\left(\frac{-c + i\sqrt{4mk - c^2}}{2m}t\right).$$

According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
x(t) &= C_1 \exp\left(\frac{-c - i\sqrt{4mk - c^2}}{2m}t\right) + C_2 \exp\left(\frac{-c + i\sqrt{4mk - c^2}}{2m}t\right) \\
&= C_1 \exp\left(-\frac{c}{2m}t\right) \exp\left(-i\frac{\sqrt{4mk - c^2}}{2m}t\right) + C_2 \exp\left(-\frac{c}{2m}t\right) \exp\left(i\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&= \exp\left(-\frac{c}{2m}t\right) \left[C_1 \exp\left(-i\frac{\sqrt{4mk - c^2}}{2m}t\right) + C_2 \exp\left(i\frac{\sqrt{4mk - c^2}}{2m}t\right)\right] \\
&= \exp\left(-\frac{c}{2m}t\right) \left[C_1 \left(\cos\frac{\sqrt{4mk - c^2}}{2m}t - i\sin\frac{\sqrt{4mk - c^2}}{2m}t\right)\right. \\
&\qquad \left. + C_2 \left(\cos\frac{\sqrt{4mk - c^2}}{2m}t + i\sin\frac{\sqrt{4mk - c^2}}{2m}t\right)\right] \\
&= \exp\left(-\frac{c}{2m}t\right) \left[(C_1 + C_2)\cos\frac{\sqrt{4mk - c^2}}{2m}t + (-iC_1 + iC_2)\sin\frac{\sqrt{4mk - c^2}}{2m}t\right] \\
&= \exp\left(-\frac{c}{2m}t\right) \left(C_3\cos\frac{\sqrt{4mk - c^2}}{2m}t + C_4\sin\frac{\sqrt{4mk - c^2}}{2m}t\right)
\end{aligned}$$

Differentiate it with respect to t.

$$\begin{aligned}
\frac{dx}{dt} &= -\frac{c}{2m}\exp\left(-\frac{c}{2m}t\right)\left(C_3\cos\frac{\sqrt{4mk - c^2}}{2m}t + C_4\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&\quad + \exp\left(-\frac{c}{2m}t\right)\left(-C_3\frac{\sqrt{4mk - c^2}}{2m}\sin\frac{\sqrt{4mk - c^2}}{2m}t + C_4\frac{\sqrt{4mk - c^2}}{2m}\cos\frac{\sqrt{4mk - c^2}}{2m}t\right)
\end{aligned}$$

Apply the initial conditions to determine C_3 and C_4.

$$x(0) = C_3 = 0$$

$$\frac{dx}{dt}(0) = -\frac{c}{2m}C_3 + C_4\frac{\sqrt{4mk - c^2}}{2m} = 2.4$$

Solving this system yields

$$C_3 = 0 \quad \text{and} \quad C_4 = \frac{24m}{5\sqrt{4mk - c^2}},$$

which means the solution to the initial value problem is

$$x(t) = \frac{24m}{5\sqrt{4mk - c^2}}\exp\left(-\frac{c}{2m}t\right)\sin\frac{\sqrt{4mk - c^2}}{2m}t.$$

Therefore, plugging in $m = 2$ kg, $c = 16$ N $\cdot$ s/m, and $k = 64$ N/m,

$$x(t) = \frac{3}{5}e^{-4t}\sin 4t.$$

Below is a plot of the displacement from equilibrium (in meters) versus time (in seconds).